The Newton and Schwarzschild Geodesics Compared

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This abbreviated note compares the geodesic equations of Newton and Schwarzschild in light of the Planck vacuum theory. The resulting equations are expressed in terms of the n-factor n_r which represents the relative stress the attractor mass M exerts on the Planck vacuum and the curved spacetime of General Relativity. The test mass m is assumed to be vanishingly small compared to this attractor mass.

The Newton geodesic equations are [1, p.74]

$$\ddot{r} + \frac{MG}{r^2} - \frac{\tilde{l}^2}{r^3} = 0 \tag{1}$$

and

$$\tilde{l} = \frac{l}{m} = r^2 \dot{\phi}\,. \tag{2}$$

Using

$$\frac{MG}{r^2} = \frac{(c^2/r)(Mc^2/r)}{c^4/G} = \frac{c^2}{r}\frac{Mc^2/r}{m_* c^2/r_*} = \frac{c^2}{r} n_r \tag{3}$$

where

$$n_r = \frac{Mc^2/r}{m_* c^2/r_*} \tag{4}$$

leads to

$$\ddot{r} = -\frac{c^2}{r}\, n_r + \frac{\tilde{l}^2}{r^3}\,. \tag{5}$$

The Schwarzschild geodesic equations are [2, p.92]

$$\frac{d^2 r}{ds^2} - \frac{\nu'}{2}\left(\frac{dr}{ds}\right)^2 - re^{\nu}\left(\frac{d\phi}{ds}\right)^2 + \frac{\nu' e^{2\nu}}{2}\left(\frac{dt}{ds}\right)^2 c^2 = 0 \tag{6}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0 \tag{7}$$

$$\frac{d^2 t}{ds^2} + \nu' \frac{dr}{ds}\frac{dt}{ds} = 0 \tag{8}$$

where

$$e^{\nu} = 1 - \frac{2MG}{c^2 r} = 1 - 2n_r \tag{9}$$

and $\nu' = d\nu/dr$.

Applying (8) in the usual way to (6) yields

$$\ddot{r} - \dot{\nu}\dot{r} - \frac{\nu' \dot{r}^2}{2} - re^{\nu}\dot{\phi}^2 + \frac{\nu' e^{2\nu}}{2}\, c^2 = 0 \tag{10}$$

where $\dot{\nu} = d\nu/dt$. Applying (8) to (7) leads to

$$\tilde{l} = \frac{r^2\dot{\phi}}{e^\nu} = \frac{r^2\dot{\phi}}{1-2n_r} \tag{11}$$

where $\tilde{l}$ is the angular momentum per unit test-particle rest mass m.

Inserting (11) into (10) leads to the geodesic

$$\ddot{r} - \dot{\nu}\dot{r} - \frac{\nu'\dot{r}^2}{2} - \frac{e^{3\nu}}{r^3}\,\tilde{l}^{\,2} + \frac{\nu' e^{2\nu}}{2}\,c^2 = 0\,. \tag{12}$$

Using (9) to calculate ν' and $\dot{\nu}$ for (12) produces

$$\ddot{r} - \frac{3}{1-2n_r}\left(\frac{\dot{r}}{c}\right)^2\frac{c^2 n_r}{r} + (1-2n_r)\frac{c^2 n_r}{r} - \left(\frac{1-2n_r}{r}\right)^3\tilde{l}^{\,2} = 0\,. \tag{13}$$

In summary, the geodesic equations for Newton are:

$$\ddot{r} = -\frac{c^2 n_r}{r} + \frac{\tilde{l}^2}{r^3} \tag{14}$$

and

$$\tilde{l} = \frac{l}{m} = r^2\dot{\phi} \tag{15}$$

while those for Schwarzschild are:

$$\begin{aligned}\ddot{r} = &-(1-2n_r)\frac{c^2 n_r}{r} + \left(\frac{1-2n_r}{r}\right)^3\tilde{l}^{\,2}\\ &+\frac{3}{1-2n_r}\left(\frac{\dot{r}}{c}\right)^2\frac{c^2 n_r}{r} = 0\end{aligned} \tag{16}$$

and

$$\tilde{l} = \frac{r^2\dot{\phi}}{1-2n_r}\,. \tag{17}$$

Note the difference between the Newton (15) and the Schwarzschild (17) constants of motion. For $\dot{r} = 0$ (16) becomes

$$\ddot{r} = -(1-2n_r)\frac{c^2 n_r}{r} + \left(\frac{1-2n_r}{r}\right)^3\tilde{l}^{\,2} \tag{18}$$

allowing the "static gravitational fields" (14) and (18) from the Newton and Schwarzschild geodesics to be compared.

References

[1] Goldstein H., Poole P.P. Jr., Safko J.L.: Classical mechanics. Third edition, Pearson Education, LPE, 2000.

[2] Narlikar J.V.: An introduction to cosmology. Third edition, Cambridge Univ. Press, Cambridge, UK, 2002.